ASYMPTOTIC SOLUTION OF THE PROBLEM OF AN ELASTIC LAYER

(ASIMPTOTICHESKOE RESHENIE ZADACHI OB UPRUGOM SLOE)

PMM Vol.30, № 4, 1966, pp.754-759

S.S. DYMKOV
(Leningrad)

(Received July 12, 1965)

Equations of the theory of elasticity in displacements are solved for a layer of thickness h by utilizing Fourier transformations. An asymptotic of the solution is obtained in the parameter h, where the order of the asymptotic depends on the differential properties of the functions describing the volume forces and the external stresses on the layer endfaces. Equations reducing the solution of the three-dimensional problem to the solution of a chain of two-dimensional problems are derived.

A great number of works is devoted to the solution of the problem of an elastic layer. The fundamental ones are cited in a short survey presented in the book by Lur'e [1] in which are also given original researches utilizing the symbolic method. The Fourier transform is applied herein, which results in intermediate formulas of substantially the same form as in [1]. However, use of the Fourier transform permits a more detailed analysis of the obtained expressions by reliance on the apparatus of generalized functions. By this means the asymptotic of the solution of the problem has successfully been obtained, and the solution has also been reduced, as is customary in such kind of three-dimensional problems, to the solution of a chain of two-dimensional problems.

Let us consider an elastic layer of thickness h whose middle plane coincides with the $z = 0$ plane, in an x, y, z Cartesian coordinate system. Let $\mathbf{n}$ be the unit vector normal to the middle plane, directed in the positive z direction. Let us decompose the displacement vector of points of the layer into the vector $\mathbf{u}$ parallel to the middle plane, and the vector $w\mathbf{n}$ perpendicular to it, where w is a scalar function. Analogously, let us decompose the volume force referred to unit volume of the layer into a vector $\mathbf{M}$ parallel to the middle plane and a vector $N\mathbf{n}$ perpendicular to it. Instead of the z coordinate it is convenient to consider the nondimensional coordinate

$$\zeta = 2z / h \qquad (-1 \leqslant \zeta \leqslant +1)$$

Utilizing the notation introduced, the equations of the theory of elasticity in displacements may be written as

$$\frac{1}{4} h^2 \Big(\Delta \mathbf{u} + \frac{1}{1-2\nu} \operatorname{grad} \operatorname{div} \mathbf{u}\Big) + \frac{\partial^2 \mathbf{u}}{\partial \zeta^2} + \frac{1}{2} h \frac{1}{1-2\nu} \operatorname{grad} \frac{\partial w}{\partial \zeta} + \frac{1}{4} h^2 \frac{1}{G} \mathbf{M} = 0$$

$$\frac{1}{4} h^2 \Delta w + 2 \frac{1-\nu}{1-2\nu} \frac{\partial^2 w}{\partial \zeta^2} + \frac{1}{2} h \frac{1}{1-2\nu} \operatorname{div} \frac{\partial \mathbf{u}}{\partial \zeta} + \frac{1}{4} h^2 \frac{1}{G} N = 0 \qquad (1)$$

where G is the shear modulus, and ν is the Poisson coefficient. Here, and henceforth throughout, the differential operators Δ , grad and div act only on the variables x , y . Let us turn to conditions on the layer surfaces. The external force acting on the upper surface of the layer $\zeta = +1$ may be expanded into the tangential force $\mathbf{t}_+$ per unit area distributed over the upper surface, and the force $p_+\mathbf{n}$ acting along the normal, which is also referred to unit area of this surface. The analogous external forces acting on the lower surface of the layer $\zeta = -1$ are denoted by $\mathbf{t}_-$ and $p_-\mathbf{n}$. In the conventional notation the conditions on the layer endfaces are

$$\frac{1}{2}h\operatorname{grad} w+\frac{\partial \mathbf{u}}{\partial \zeta}=\frac{1}{2}h\frac{1}{G}\begin{cases}\ \ \mathbf{t}_+ & (\zeta=+1)\\ -\mathbf{t}_- & (\zeta=-1)\end{cases}$$
$$\frac{1}{2}h\operatorname{div}\mathbf{u}+\frac{1-\nu}{\nu}\frac{\partial w}{\partial \zeta}=\frac{1}{2}h\frac{1-2\nu}{2G\nu}\begin{cases}\ \ p_+ & (\zeta=+1)\\ -p_- & (\zeta=-1)\end{cases} \tag{2}$$

Let us consider separately the cases of nonsymmetric and symmetric loading of the layer on the endfaces in the absence of volume forces, and the case of a layer subjected to volume forces but free of stresses in the endfaces. Let us put

$$\mathbf{T}_+={}^1/_2(\mathbf{t}_++\mathbf{t}_-),\qquad \mathbf{T}_-={}^1/_2(\mathbf{t}_+-\mathbf{t}_-),\qquad P_+={}^1/_2(p_++p_-),\qquad P_-={}^1/_2(p_+-p_-)$$

For the nonsymmetric loading (Problem A) we shall solve Equation

$$\frac{1}{4}h^2\left(\Delta\mathbf{u}+\frac{1}{1-2}\operatorname{grad}\operatorname{div}\mathbf{u}\right)+\frac{\partial^2\mathbf{u}}{\partial\zeta^2}+\frac{1}{2}h\frac{1}{1-2\nu}\operatorname{grad}\frac{\partial w}{\partial\zeta}=0$$
$$\frac{1}{4}h^2\Delta w+2\,\frac{1-\nu}{1-2\nu}\frac{\partial^2 w}{\partial\zeta^2}+\frac{1}{2}h\frac{1}{1-2\nu}\operatorname{div}\frac{\partial\mathbf{u}}{\partial\zeta}=0 \tag{3}$$

with conditions on the endfaces

$$\frac{1}{2}h\operatorname{grad} w+\frac{\partial\mathbf{u}}{\partial\zeta}=\frac{1}{2}h\frac{1}{G}\mathbf{T}_-,\quad \frac{1}{2}h\operatorname{div}\mathbf{u}+\frac{1-\nu}{\nu}\frac{\partial w}{\partial\zeta}=\pm\frac{1}{2}h\frac{1-2\nu}{2G\nu}P_+(\zeta=\pm 1) \tag{4}$$

For the symmetric loading (Problem B) conditions (4) are replaced by

$$\frac{1}{2}h\operatorname{grad} w+\frac{\partial\mathbf{u}}{\partial\zeta}=\pm\frac{1}{2}h\frac{1}{G}\mathbf{T}_+,\qquad \frac{1}{2}h\operatorname{div}\mathbf{u}+\frac{1-\nu}{\nu}\frac{\partial w}{\partial\zeta}=\frac{1}{2}h\frac{1-2\nu}{2G\nu}P_-\quad(\zeta=\pm 1)$$

The case of volume forces (Problem C) reduces to the solution of (1) with homogeneous conditions on the endfaces

$$\frac{1}{2}h\operatorname{grad} w+\frac{\partial\mathbf{u}}{\partial\zeta}=0,\qquad \frac{1}{2}h\operatorname{div}\mathbf{u}+\frac{1-\nu}{\nu}\frac{\partial w}{\partial\zeta}=0\quad(\zeta=\pm 1)$$

The solution of the original problem (1), (2) is given by the sum of the solutions of Problems A, B and C .

Let the vector function $\mathbf{M}$ and scalar N admit of the expansions

$$\mathbf{M}=\sum_{n=0}^{\infty}\frac{1}{n!}({}^1/_2\,h\zeta)^n\mathbf{M}_0^{(n)},\qquad N=\sum_{n=0}^{\infty}\frac{1}{n!}({}^1/_2\,h\zeta)^n N_0^{(n)}$$
$$\mathbf{M}_0^{(n)}=\left.\frac{\partial^n\mathbf{M}(x,y,z)}{\partial z^n}\right|_{z=0},\qquad N_0^{(n)}=\left.\frac{\partial^n N(x,y,z)}{\partial z^n}\right|_{z=0} \tag{5}$$

It is assumed in all the subsequent exposition that the vector functions $\mathbf{M}_0^{(n)}$, $\mathbf{T}_+$, $\mathbf{T}_-$ and the scalars $N_0^{(n)}$, P_+, P_-, which depend only on the variables x, y, admit of Fourier integral representations in these variables. Let $\mathbf{r}$ and $\mathbf{k}$ denote two radius-vectors with components x, y and ξ, η, respectively. We denote the Fourier transform of the function $f(\mathbf{r})$ by $f^*(\mathbf{k})$. Hence

$$f^*(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r})\, e^{i\mathbf{kr}}\, dx\, dy, \qquad f(\mathbf{r}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(\mathbf{k})\, e^{-i\mathbf{kr}}\, d\xi\, d\eta$$

Let us examine the solution of Problem A in more detail. Let us apply the Fourier transform in the x, y variables to (3) and the endface conditions (4). The variable ζ should hence be considered a parameter. The transformed equations and endface conditions may be written thus

$$\begin{gathered}\frac{\partial^2 \mathbf{u}^*}{\partial \zeta^2} + i\frac{1}{2}h\frac{1}{1-2\nu}\frac{\partial w^*}{\partial \zeta}\mathbf{k} - \frac{1}{4}h^2\left[k^2\mathbf{u}^* + \frac{1}{1-2\nu}(\mathbf{u}^*\mathbf{k})\mathbf{k}\right] = 0 \\ \frac{\partial^2 w^*}{\partial \zeta^2} + i\frac{1}{2}h\frac{1}{2(1-\nu)}\frac{\partial(\mathbf{u}^*\mathbf{k})}{\partial \zeta} - \frac{1}{4}h^2k^2\frac{1-2\nu}{2(1-\nu)}w^* = 0\end{gathered} \tag{6}$$

$$\begin{gathered}\frac{\partial \mathbf{u}^*}{\partial \zeta} + i\frac{1}{2}hw^*\mathbf{k} = \frac{1}{2}h\frac{1}{G}\mathbf{T}_-{}^* \\ \frac{\partial w^*}{\partial \zeta} + i\frac{1}{2}h\frac{\nu}{1-\nu}\mathbf{u}^*\mathbf{k} = \pm\frac{1}{2}h\frac{1-2\nu}{2G(1-\nu)}P_+{}^*\end{gathered} \qquad (\zeta = \pm 1) \tag{7}$$

Here k is the modulus of the vector $\mathbf{k}$ ($k = |\mathbf{k}|$). Let us introduce the vector $\mathbf{Kn}\times\mathbf{k}$, which is, exactly as is $\mathbf{k}$, parallel to the middle plane and equal to the vector $\mathbf{k}$ in absolute value. Therefore, the vectors $\mathbf{u}^*$ and $\mathbf{T}_-^*$ may be decomposed into single linearly independent vectors $k^{-1}\mathbf{k}$ and $k^{-1}\mathbf{K}$. Let $u_\mathbf{k}^*$ and $u_\mathbf{K}^*$ denote the components of the decomposition of the vector w^*, and similarly for the components of the decomposition of the vector $\mathbf{T}_-^*$. Multiplying scalarly the first equation of (6) by $k^{-1}\mathbf{k}$ and $k^{-1}\mathbf{K}$ successively, we replace them by a system of three equations in the scalar functions $u_\mathbf{k}^*$, $u_\mathbf{K}^*$ and w^*

$$\begin{gathered}\frac{\partial^2 u_\mathbf{k}^*}{\partial \zeta^2} + i\frac{1}{2}hk\frac{1}{1-2\nu}\frac{\partial w^*}{\partial \zeta} - \frac{1}{4}h^2k^2\frac{2(1-\nu)}{1-2\nu}u_\mathbf{k}^* = 0 \\ \frac{\partial^2 w^*}{\partial \zeta^2} + i\frac{1}{2}hk\frac{1}{2(1-\nu)}\frac{\partial u_\mathbf{k}^*}{\partial \zeta} - \frac{1}{4}h^2k^2w^* = 0, \qquad \frac{\partial^2 u\mathbf{K}^*}{\partial \zeta^2} - \frac{1}{4}h^2k^2u\mathbf{K}^* = 0\end{gathered} \tag{8}$$

The endface conditions become

$$\begin{gathered}\frac{\partial u_\mathbf{k}^*}{\partial \zeta} + i\frac{1}{2}hkw^* = \frac{1}{2}h\frac{1}{G}T_{-\mathbf{k}}{}^*, \qquad \frac{\partial u\mathbf{K}^*}{\partial \zeta} = \frac{1}{2}h\frac{1}{G}T_{-\mathbf{K}}{}^* \\ \frac{\partial w^*}{\partial \zeta} + i\frac{1}{2}hk\frac{\nu}{1-\nu}u_\mathbf{k}^* = \pm\frac{1}{2}h\frac{1-2\nu}{2G(1-\nu)}P_+{}^* \qquad (\zeta = \pm 1)\end{gathered} \tag{9}$$

We shall consider (8) as a system of ordinary differential equations in the argument ζ. The solution of this system taking account of the boundary conditions (9) may be written as follows:

$$\begin{aligned} u_\mathbf{k}^* = \frac{1}{Gk}\Bigg\{ &-\frac{[{}^1\!/_2\, hk\cosh{}^1\!/_2\, hk - 2(1-\nu)\sinh{}^1\!/_2\, hk]\sinh{}^1\!/_2\, hk\zeta - \sinh{}^1\!/_2\, hk\,{}^1\!/_2\, hk\zeta\cosh{}^1\!/_2\, hk\zeta}{\sinh hk - hk}T_{-\mathbf{k}}{}^* + \\ &+ \frac{[{}^1\!/_2\, hk\sinh{}^1\!/_2\, hk - (1-2\nu)\cosh{}^1\!/_2\, hk]\sinh{}^1\!/_2\, hk\zeta - \cosh{}^1\!/_2\, hk\,{}^1\!/_2\, hk\zeta\cosh{}^1\!/_2\, hk\zeta}{\sinh hk - hk}iP_+{}^*\Bigg\} \end{aligned} \tag{10}$$

$$\begin{aligned} w^* = \frac{1}{Gk}\Bigg\{ &-\frac{\sinh{}^1\!/_2\, hk\,{}^1\!/_2\, hk\zeta\sinh{}^1\!/_2\, hk\zeta - [{}^1\!/_2\, hk\cosh{}^1\!/_2\, hk + (1-2\nu)\sinh{}^1\!/_2\, hk]\cosh{}^1\!/_2\, hk\zeta}{\sinh hk - hk}iT_{-\mathbf{k}}{}^* - \\ &-\frac{\cosh{}^1\!/_2\, hk\,{}^1\!/_2\, hk\zeta\sinh{}^1\!/_2\, hk\zeta - [{}^1\!/_2\, hk\sinh{}^1\!/_2\, hk + 2(1-\nu)\cosh{}^1\!/_2\, hk]\cosh{}^1\!/_2\, hk\zeta}{\sinh hk - hk}P_+{}^*\Bigg\} \end{aligned}$$

$$u_\mathbf{K}^* = \frac{1}{Gk}\frac{\sinh{}^1\!/_2\, hk\zeta}{\cosh{}^1\!/_2\, hk}T_{-\mathbf{K}}{}^*$$

The coefficients of the functions $\mathbf{T}_-^*$ and $\mathbf{P}_+^*$ in (10) are analytic functions of h and may be expanded in Laurent series in the neighborhood of the point $h = 0$. Such an expansion leads to the following expressions:

$$\mathbf{u}^* = \frac{\zeta}{G}\left\{-\frac{1}{h^2}\frac{6(1-\nu)}{k^4} iP_+{}^*\mathbf{k} + \frac{1}{h}\frac{3(1-\nu)}{k^4}(\mathbf{T}_-{}^*\mathbf{k})\mathbf{k} + \frac{3\nu-(2-\nu)\zeta^2}{4k^2} iP_+{}^*\mathbf{k} + O(h)\right\}$$

$$w^* = \frac{1}{G}\left\{\frac{1}{h^3}\frac{12(1-\nu)}{k^4}P_+{}^* + \frac{1}{h^2}\frac{6(1-\nu)}{k^4} i(\mathbf{T}_-{}^*\mathbf{k}) + \frac{1}{h}\frac{3[(2-\nu)-\nu\zeta^2]}{2k^2}P_+{}^* + \frac{(2-\nu)-3\nu\zeta^2}{4k^2} i(\mathbf{T}_-{}^*\mathbf{k}) + O(h)\right\} \quad (11)$$

where the series are written down to terms of order h . In order to obtain the decomposition of $\mathbf{u}^*$ it has here been taken into account that

$$\mathbf{u}^* = u_{\mathbf{k}}{}^* k^{-1}\mathbf{k} + u_{\mathbf{K}}{}^* k^{-1}\mathbf{K}$$

where, by virtue of the last formula in (10), the second member on the right may be written as

$$u_{\mathbf{K}}{}^* k^{-1}\mathbf{K} = \frac{1}{Gk}\frac{\sinh {}^1/_2 hk\zeta}{\cosh {}^1/_2 hk}[\mathbf{T}_-{}^* - k^{-2}(\mathbf{T}_-{}^*\mathbf{k})\mathbf{k}]$$

Applying the inverse Fourier transform term by term to both sides of (11), we reduce the right side to integrals of the form

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\mathbf{F}(\mathbf{r}')\mathbf{k}]\mathbf{k}k^{2m}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}d\xi\, d\eta\, dy'\, dy' = (-1)^{m+1}\operatorname{grad}\operatorname{div}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{F}(\mathbf{r}')\Delta^m\delta(\mathbf{r}-\mathbf{r}')\,dx'\,dy'$$

$$\frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')\mathbf{k}k^{2m}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}d\xi\, d\eta\, dx'\, dy' = (-1)^m\operatorname{grad}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')\Delta^m\delta(\mathbf{r}-\mathbf{r}')\,dx'\,dy' \quad (12)$$

$$\frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{F}(\mathbf{r}')\mathbf{k}k^{2m}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}d\xi\, d\eta\, dx'\, dy' = (-1)^m\operatorname{div}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{F}(\mathbf{r}')\Delta^m\delta(\mathbf{r}-\mathbf{r}')\,dx'\,dy'$$

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')k^{2m}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}d\xi\, d\eta\, dx'\, dy' = (-1)^m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')\Delta^m\delta(\mathbf{r}-\mathbf{r}')\,dx'\,dy'$$

where $\delta(\mathbf{r})$ is the Dirac function on a plane. If $m = -1$ or $m = -2$, then the fundamental singular solutions of the Laplace or biharmonic equations, respectively,

$$\Delta^{-1}\delta(\mathbf{r}) = \frac{1}{2\pi}\ln r, \qquad \Delta^{-2}\delta(\mathbf{r}) = \frac{1}{8\pi}r^2\ln r$$

may be substituted into the right sides of (12) with the result that we arrive at the customary integrals of potential type. According to the fundamental property of the δ-function, we have for $m = 0$ at all points of continuity of the function $f(\mathbf{r})$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\,dx'\,dy' = f(\mathbf{r}) \quad (13)$$

Thus, for the classical solution, if differentiability of the functions $\mathbf{T}$ and $\mathbf{P}$ is required, just the terms without positive powers of k may be separated out of the expansions. Hence, we can obtain an asymptotic up to terms of order $O(h^0)$ inclusive. Thus, performing the inverse Fourier transformation of (11) and utilizing (12) we obtain

$$\mathbf{u}(\mathbf{r},\zeta) = -\frac{\zeta}{8\pi G}\operatorname{grad}\left\{\frac{6(1-\nu)}{h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_+(\mathbf{r}')(\mathbf{r}-\mathbf{r}')^2\ln|\mathbf{r}-\mathbf{r}'|\,dx'\,dy' + \right.$$

$$+\frac{3(1-\nu)}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{T}_-(\mathbf{r}')(\mathbf{r}-\mathbf{r}')(2\ln|\mathbf{r}-\mathbf{r}'|+1)\,dx'\,dy' +$$

$$\left. + [3\nu-(2-\nu)\zeta^2]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_+(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'|\,dx'\,dy'\right\} + \frac{h\zeta}{G}\mathbf{U}$$

$$w(\mathbf{r},\zeta) = \frac{1}{8\pi G}\left\{\frac{12(1-\nu)}{h^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_+(\mathbf{r}')(\mathbf{r}-\mathbf{r}')^2\ln|\mathbf{r}-\mathbf{r}'|\,dx'\,dy' + \right.$$

$$+\frac{6(1-\nu)}{h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{T}_-(\mathbf{r}')(\mathbf{r}-\mathbf{r}')(2\ln|\mathbf{r}-\mathbf{r}'|+1)\,dx'\,dy' -$$

$$-\frac{6[(2-\nu)-\nu\zeta^2]}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_+(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'|\,dx'\,dy' -$$

$$\left. -[(2-\nu)-3\nu\zeta^2]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{T}_-(\mathbf{r}')(\mathbf{r}-\mathbf{r}')\frac{1}{(\mathbf{r}-\mathbf{r}')^2}\,dx'\,dy'\right\} + \frac{h}{G}W$$

where the functions $\mathbf{U}$ and W tend to their limiting values as $h \to 0$, which may easily be evaluated. The limiting values of $\mathbf{U}$ and W depending on the functions $\mathbf{T}_-$ and P_+ are finite if the functions $\mathbf{T}_-$ and P_+ are bounded in absolute value.

If the functions $\mathbf{T}_-$ and P_+ have derivatives up to the order $2m$, then terms up to the order of $2m$ in h may be separated out in the Laurent expansions. In the appropriate integrals of (12) we can transfer from the δ-functions in the right sides to iteration of the Laplace operator in the functions $\mathbf{T}_-$ or P_+, thereby arriving at formulas of the form (13). Hence, an asymptotic of correspondingly higher order is obtained. Reasoning, analogous to the presented above, may be repeated relative to the remaining terms, where boundedness of some differential operators of the functions $\mathbf{T}_-$ and P_+ in absolute value is required for boundedness of their limiting values.

By using the obtained expressions the solution of the problem on an elastic layer may be reduced to the solution of a chain of two-dimensional problems. If the solution is represented as a power series in h **(14)**

$$w = \frac{1}{h^3}w_{-3} + \frac{1}{h^2}w_{-2} + \frac{1}{h}w_{-1} + w_0 + hw_1 + \ldots,\quad \mathbf{u} = \frac{1}{h^2}\mathbf{u}_{-2} + \frac{1}{h}\mathbf{u}_{-1} + \mathbf{u}_0 + h\mathbf{u}_1 + \ldots$$

then we can put

$$w_{-3} = W_{-3},\quad w_{-2} = W_{-2},\quad w_{-1} = W_{-1} + \frac{1}{8}\frac{\nu}{1-\nu}\zeta^2\Delta W_{-3},\quad W_0 = w_0 + \frac{1}{8}\frac{\nu}{1-\nu}\zeta^2\Delta W_{-2},\ldots$$

$$u_{-2} = -\frac{\zeta}{2}\operatorname{grad} W_{-3},\qquad u_1 = -\frac{\zeta}{2}\operatorname{grad} W_{-2}$$

$$u_0 = -\frac{\zeta}{2}\operatorname{grad}\left(W_{-1} + \frac{1}{4}\frac{1}{1-\nu}\Delta W_{-3}\right) + \frac{1}{48}\frac{2-\nu}{1-\nu}\zeta^3\operatorname{grad}\Delta W_{-3},\ldots$$

Here the functions W_{-3}, $W_{-2}, \ldots$ satisfy Equations

$$\Delta^2 W_{-3} = \frac{12(1-\nu)}{G} P_+, \qquad \Delta^2 W_{-1} = -\frac{3(2-\nu)}{2G} \Delta P_+$$
$$\Delta^2 W_{-2} = \frac{6(1-\nu)}{G} \operatorname{div} \mathbf{T}_-, \qquad \Delta^2 W_0 = -\frac{2-\nu}{4G} \Delta \operatorname{div} \mathbf{T}_-, \ldots \tag{15}$$

These equations will be valid in the sense in which the differential operations on the functions $\mathbf{T}_-$ and P_+ in their right sides may be considered. Thus, if the classical solution is sought, the functions W_i may be calculated to values of the subscript i for which the differential properties of the functions $\mathbf{T}_-$ and P_+ permit application of the operators in the right sides of the appropriate equations in (15).

The first equation in (15) agrees with the known equation of Sophie Germain in plate theory. If the Kirchhoff hypothesis is used, this equation may be deduced, which therefore corresponds to the first approximation in the obtained asymptotic expansion (14). It is interesting to note that we can arrive formally at (15) if (14) is substituted into (3) and the conditions (4) are terms in identical powers of h are equated. Equations (3) hence reduce to a recursion system of ordinary differential equations in ζ, where x and y may be considered as parameters on which the arbitrary functions W_i depend. The latter are selected so that the endface conditions would be satisfied. These conditions will be satisfied if the W_i satisfy Equations (15).

Problem B is solved completely analogously, hence, we present just the final results. For the functions $u_{\mathbf{k}}^*, w$ and $u_{\mathbf{K}}^*$ we will have

$$u_{\mathbf{k}}^* = \frac{1}{Gk}\left\{-\frac{[{}^1/_2\, hk \sinh {}^1/_2\, hk - 2(1-\nu)\cosh {}^1/_2\, hk]\cosh {}^1/_2\, hk\zeta - \cosh {}^1/_2\, hk\, {}^1/_2\, hk\zeta \sinh {}^1/_2\, hk\zeta}{\sinh hk + hk} T_{+\mathbf{k}}^* + \right.$$
$$\left. + \frac{[{}^1/_2\, hk \cosh {}^1/_2\, hk - (1-2\nu)\sinh {}^1/_2 hk]\cosh {}^1/_2 hk\zeta - \sinh {}^1/_2 hk\, {}^1/_2 hk\zeta \sinh {}^1/_2 hk\zeta}{\sinh hk + hk} iP_-^*\right\} =$$
$$= \frac{1}{Gk}\left[\frac{1}{h}\,\frac{1-\nu}{k} T_{+\mathbf{k}}^* + \frac{1}{2}\nu i P_-^* + O(h)\right]$$

$$w^* = \frac{1}{Gk}\left\{-\frac{\cosh {}^1/_2 hk\, {}^1/_2 hk\zeta \cosh {}^1/_2 hk\zeta - [{}^1/_2 hk \sinh {}^1/_2 hk + (1-2\nu)\cosh {}^1/_2 hk]\sinh {}^1/_2 hk\zeta}{\sinh kk + hk} iT_{+\mathbf{k}}^* - \right.$$
$$\left. - \frac{\sinh {}^1/_2 hk\, {}^1/_2 hk\zeta \cosh {}^1/_2 hk\zeta - [{}^1/_2 hk \cosh {}^1/_2 hk + 2(1-\nu)\sinh {}^1/_2 hk]\sinh {}^1/_2 hk\zeta}{\sinh hk + hk} P_-^* - \right\} =$$
$$= \frac{\zeta}{Gk}\left[-\frac{1}{2}\nu i T_{+\mathbf{k}}^* + O(h)\right]$$
$$u_{\mathbf{K}}^* = \frac{1}{Gk}\,\frac{\cosh {}^1/_2 hk\zeta}{\sinh {}^1/_2\, hk} T_{+\mathbf{K}}^* = \frac{1}{Gk}\left[\frac{1}{h}\,\frac{2}{k} T_{+\mathbf{K}}^* + O(h)\right]$$

and therefore

$$\mathbf{u}(\mathbf{r}, \zeta) = -\frac{1}{\pi G}\frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{T}_+(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'|\,dx'\,dy' +$$
$$+ \frac{1}{4\pi G}\operatorname{grad}\left[\frac{1+\nu}{2h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{T}_+(\mathbf{r}')(\mathbf{r}-\mathbf{r}')(2\ln|\mathbf{r}-\mathbf{r}'|+1)\,dx'\,dy' - \right.$$
$$\left. - \nu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_-(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'|\,dx'\,dy'\right] + O(h)$$
$$w(\mathbf{r}, \zeta) = \frac{\zeta}{4\pi G}\nu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{T}_+(\mathbf{r}')(\mathbf{r}-\mathbf{r}')\frac{1}{(\mathbf{r}-\mathbf{r}')^2}\,dx'\,dy' + O(h)$$

Finally, if we put

$$\mathbf{u}=\frac{1}{h}\mathbf{u}_{-1}+\mathbf{u}_0+h\mathbf{u}_1+\ldots,\qquad w=w_0+hw_1+\ldots$$

$$\left(\mathbf{u}_{-1}=\mathbf{U}_{-1},\ \mathbf{u}_0=\mathbf{U}_0,\ldots,w_0=-\frac{1}{2}\zeta\frac{\nu}{1-\nu}\operatorname{div}\mathbf{U}_{-1},\ldots\right),$$

then, to determine the functions $\mathbf{U}_i$ we obtain Equations

$$\Delta\mathbf{U}_{-1}+\frac{1+\nu}{1-\nu}\operatorname{grad}\operatorname{div}\mathbf{U}_{-1}=-\frac{2}{G}\mathbf{T}_{\perp}$$

$$\Delta\mathbf{U}_0+\frac{1+\nu}{1-\nu}\operatorname{grad}\operatorname{div}\mathbf{U}_0=-\frac{1}{G}\frac{\nu}{1-\nu}\operatorname{grad}P_{-},\ldots$$

In solving Problem C we must deal with the inhomogeneous equations (1). This does not introduce any difficulties in principle, however, the computation is considerably more tedious. The final results may be written as

$$w=\frac{1}{h^2}w_{-2}+\frac{1}{h}w_{-1}+w_0+hw_1+\ldots\qquad \mathbf{u}=\frac{1}{h}\mathbf{u}_{-1}+\mathbf{u}_0+h\mathbf{u}_1+\ldots$$

$$w_{-2}=W_{-2},\qquad w_{-1}=W_{-1},\qquad w_0=W_0+\frac{1}{8}\zeta^2\frac{\nu}{1-\nu}\Delta W_{-2},\ldots$$

$$u_{-1}=-\frac{1}{2}\zeta\operatorname{grad}W_{-2},\qquad \mathbf{u}_0=\mathbf{U}_0-\frac{1}{2}\zeta\operatorname{grad}W_{-1},\ldots$$

and finally the equations to determine $\mathbf{U}_i$, W_i are

$$\Delta\mathbf{U}_0+\frac{1+\nu}{1-\nu}\operatorname{grad}\operatorname{div}\mathbf{U}_0=-\frac{1}{G}\mathbf{M}_0,\ldots$$

$$\Delta^2W_{-2}=\frac{6(1-\nu)}{G}N_0,\ \Delta^2W_{-1}=0,\ \Delta^2W_0=-\frac{6-\nu}{4G}\Delta N_0+\frac{1-\nu}{2G}\operatorname{div}\mathbf{M}'_0+\frac{1-\nu}{4G}N_0'',\ldots$$

where $\mathbf{M}_0^{(n)}$ and $N_0^{(n)}$ are determined according to (5).

BIBLIOGRAPHY

1. Lur'e, A.I., Prostranstvennye zadachi teorii uprugosti (Three-Dimensional Problems of the Theory of Elasticity). M., Gostekhizdat, 1965.

Translated by M.D.F.